The Schröder - Bernstein Problem for modules

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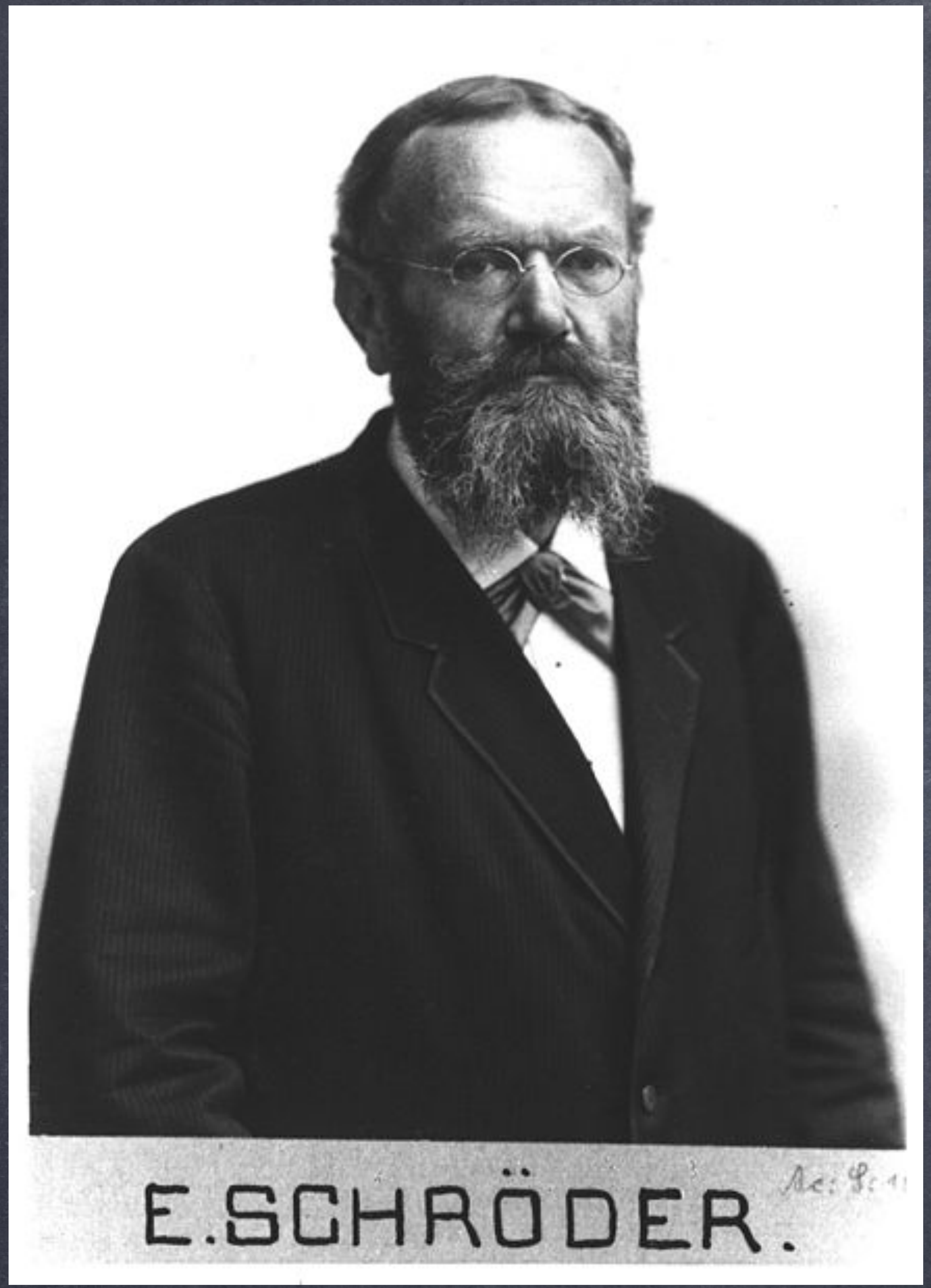
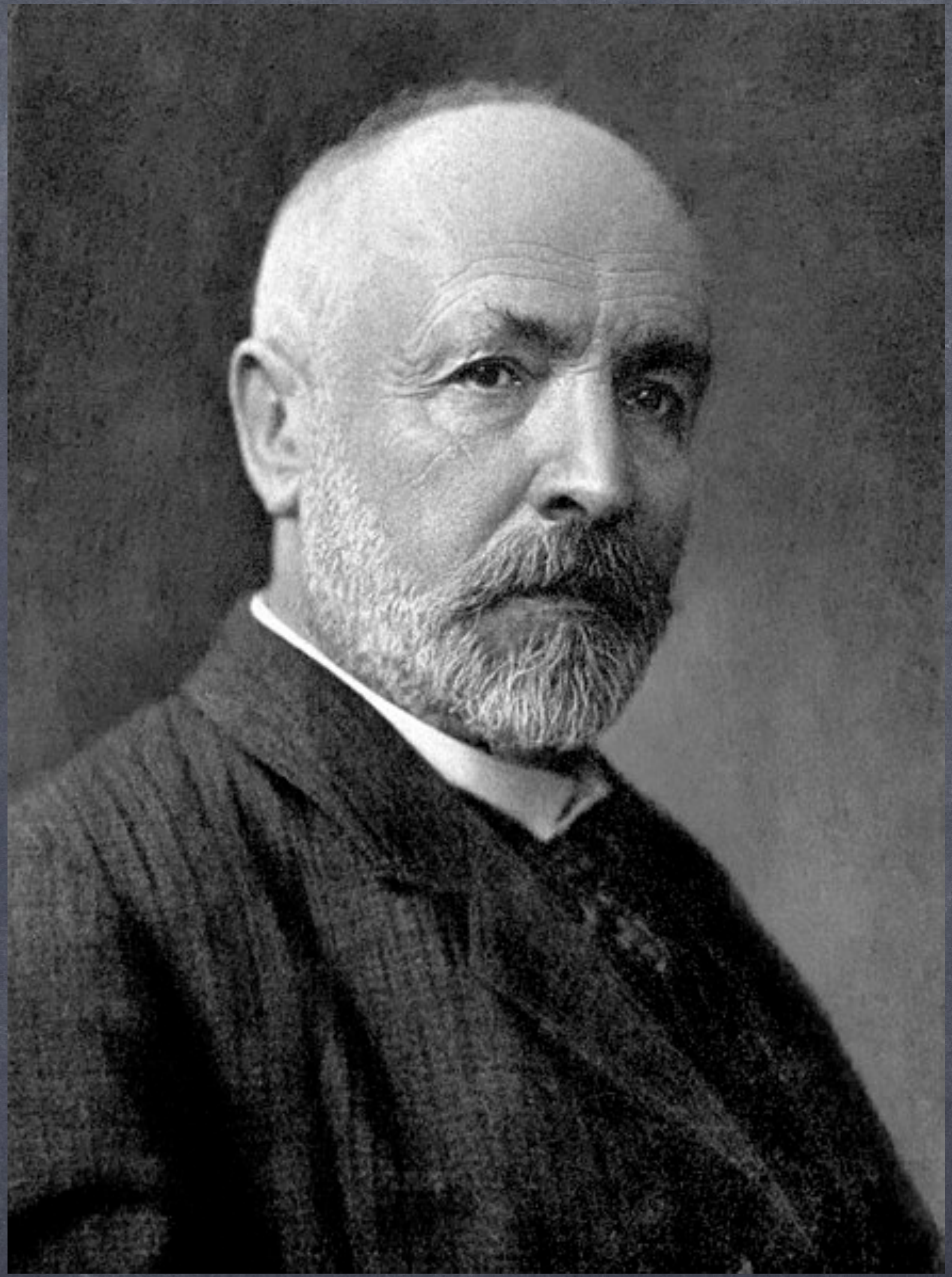

E.SCHRÖDER.



If A and B are two sets such that there is a one-one function from A into B and a one-one function from B into A then there is a bijective map between A and B.

This type of problem where one asks if two mathematical objects A and B which are similar in some sense to a part of each other are also similar themselves is usually called the Schröder-Bernstein problem.

Schröder-Bernstein problem for groups

$G_1 \xhookrightarrow{\text{mono.}} G_2$, $G_2 \xhookrightarrow{\text{mono.}} G_1$

Is $G_1 \cong G_2$?

No.

Schröder-Bernstein problem for topological spaces:

$$X_1 \xrightarrow[\text{cont.}]{\text{one-one}} X_2, \quad X_2 \xrightarrow[\text{cont.}]{\text{one-one}} X_1$$

Is $X_1 \cong X_2$?

No.

Let B_1 and B_2 be two Banach spaces such that each one is a complemented subspace of the other. Is $B_1 \cong B_2$?

No. (W. T. Gowers, 1996)

Schröder-Bernstein problem for modules:

Let M and N be modules such that $M \xrightarrow{\text{mono.}} N$, $N \xrightarrow{\text{mono.}} M$.

Is $M \cong N$?

No. $M = \bigoplus_{i \geq 1} \mathbb{Z}_{2^i}$, $N = \bigoplus_{i \geq 1} \mathbb{Z}_{4^i}$

Bumby, 1965.

- Schröder-Bernstein problem has a positive answer for modules that are invariant under endomorphisms of their injective envelopes (i.e. quasi-injective).

Müller and Rizvi, 1983:

- The Schröder - Bernstein problem has a positive answer for the class of continuous modules.

Guil Asensio, Kaleboğaz, S. (2017)

- We obtained a positive answer for the Schröder-Bernstein problem for modules invariant under endomorphisms of their general envelopes under some mild conditions.

Guil Asensio, Kaleboğaz, S. (2017)

- The Schröder-Bernstein problem has a positive answer for modules that are invariant only under automorphisms of their injective envelopes or pure-injective envelopes.

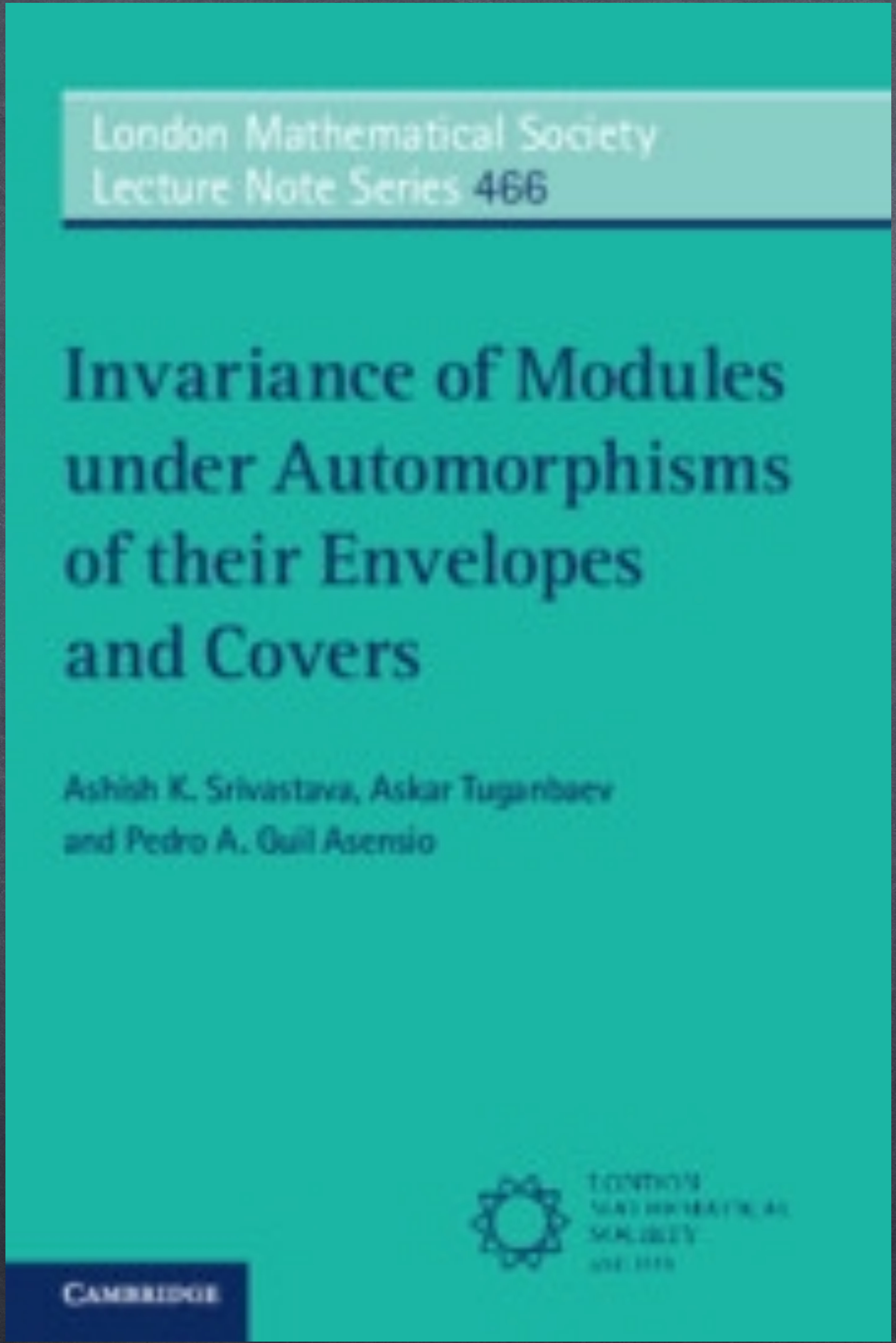
Let $\mathcal{X}$ be a class of right R-modules closed under isomorphisms and direct summands.

An $\mathcal{X}$-preenvelope of a module M is a homomorphism $u : M \longrightarrow X$, $X \in \mathcal{X}$ such that any $g : M \longrightarrow X'$, $X' \in \mathcal{X}$ factors through u.

A preenvelope $u : M \longrightarrow X$ is called an $\mathcal{X}$-envelope if it is minimal in the sense that any $h : X \longrightarrow X$ such that $h \circ u = u$ must be an automorphism.

A module M having a monomorphic $\mathcal{X}$-envelope $u: M \longrightarrow X(M)$ is said to be $\mathcal{X}$-automorphism invariant (resp., $\mathcal{X}$-endomorphism invariant) if for any automorphism (resp., endomorphism) $\varphi: X(M) \longrightarrow X(M)$, $\exists\ f: M \longrightarrow M$

$$\begin{tikzcd} M \arrow[r, dashed, "f"] \arrow[d, "u"'] & M \arrow[d, "u"] \\ X(M) \arrow[r, "\varphi"] & X(M) \end{tikzcd} \qquad u \circ f = \varphi \circ u$$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Invariance of Modules under Automorphisms of their Envelopes and Covers
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A homomorphism $u: N \longrightarrow M$ is called $\mathcal{X}$- Strongly pure monomorphism if any homomorphism $f: N \longrightarrow X$ with $X \in \mathcal{X}$ extends to a homomorphism $g: M \longrightarrow X$ such that $g \circ u = f$.

$$\begin{tikzcd} N \arrow[r, "u"] \arrow[d, "f"'] & M \arrow[dl, dashed, "g"] \\ X & \end{tikzcd}$$

- A submodule N of M is called an $\mathcal{X}$-strongly pure submodule if $i : N \longrightarrow M$ is an $\mathcal{X}$-strongly pure monomorphism.

Given a module M,
$\operatorname{add}[M] :=$ the class of all direct summands of finite direct sums of copies of M.

- M is called $\mathcal{X}$-strongly purely closed if any direct limit of splitting mono. among objects in $\operatorname{add}[M]$ is $\mathcal{X}$-strongly pure mono.

For example,

- $\mathcal{X} :=$ class of all (pure-) injective modules
  Any module is $\mathcal{X}$-strongly purely closed.

- $(\mathcal{F}, \mathcal{C})$, a cotorsion pair
  Any object in $\mathcal{F} \cap \mathcal{C}$ is $\mathcal{C}$-strongly purely closed.

Theorem: Let $M \in \mathcal{X}$ be an $\mathcal{X}$-strongly purely closed module and $N \in \mathcal{X}$, an $\mathcal{X}$-strongly pure submodule of M.
If $\exists$ an $\mathcal{X}$-strongly pure monomorphism $u: M \longrightarrow N$, then $M \cong N$.

Corollary:

1. If E is (pure-) injective and E', (pure-) injective (pure-) submodule of E such that $\exists$ a monomorphism $u: E \longrightarrow E'$, then $E \cong E'$.

2. If E is a flat cotorsion module and E', a pure submodule of E such that E' is also flat cotorsion and there exists a pure monomorphism $u: E \longrightarrow E'$, then $E \cong E'$.

Theorem: Let M, N be two $\mathcal{X}$- endomorphism invariant modules with monomorphic $\mathcal{X}$- envelopes $v_m : M \longrightarrow X(M)$ and $v_N : N \to X(N)$. Assume that N is strongly purely closed and M is an $\mathcal{X}$- strongly pure. submodule of N. If $\exists$ an $\mathcal{X}$- strongly pure mono. $u : N \longrightarrow M$, then $M \cong N$.

Corollary:

1. (Bumby) If M and N are quasi-injective modules such that there is a monomorphism from M to N and a monomorphism from N to M, then $M \cong N$.

2. If M and N are pure-quasi-injective modules such that there is a pure monomorphism from M to N and pure monomorphism from N to M, then $M \cong N$.

3. If M and N are flat modules invariant under endomorphisms of their cotorsion envelopes such that there is a pure monomorphism from M to N and a pure monomorphism from N to M, then $M \cong N$.

Theorem: Let M and N be automorphism-invariant modules and let $f: M \longrightarrow N$ and $g: N \longrightarrow M$ be monomorphisms.

Then $M \cong N$.

Proof: By Bumby, $E(M) \cong E(N)$

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "1_M"] & f(M) \arrow[r, "u"] & N \arrow[r, "g"] & M \arrow[llll, dashed, "\varphi"] \\ M & & & \end{tikzcd}$$

$$\varphi \circ g \circ u \circ f = 1_M$$

$\Rightarrow$ $u : f(M) \to N$ splits and so $f(M)$ is a direct summand of N.

Similarly, $g(N)$ is a direct summand of M.

Let $h : E(g(N)) \longrightarrow E(f(M))$ be an isomorphism. Call $M' = h^{-1}(f(M) \cap g(N))$ and $N' = h(g(N)) \cap f(M)$

$$
\begin{tikzcd}
M' \arrow[r, "h|_{M'}"] \arrow[d, "u_{M'}"'] & N' \arrow[r, "u_{N'}"] & f(M) \arrow[dll, dashed, bend right, "\varphi"'] \\
g(N) \arrow[urr, dotted, bend right, "\psi"'] & &
\end{tikzcd}
$$

$$\psi \circ u_{M'} = u_{N'} \circ h|_{M'}$$

$$\varphi \circ u_{N'} = u_{M'} \circ h^{-1}|_{N'}$$

$$\begin{tikzcd}
M' \arrow[r, "h|_{M'}"] \arrow[d, "u_{M'}"'] & N' \arrow[r, "h^{-1}|_{N'}"] \arrow[d, "u_{N'}"] & M' \arrow[d, "u_{M'}"] \\
g(N) \arrow[r] & f(M) \arrow[r] & g(N)
\end{tikzcd}$$

$$\varphi \circ \psi \circ u_{M'} = \varphi \circ u_{N'} \circ h|_{M'} = u_{M'} \circ h^{-1}|_{N'} \circ h|_{M'} = u_{M'}$$

$$(1_{g(N)} - \varphi \circ \psi) \circ u_{M'} = 0$$

As $u_{M'}$ is monic, we deduce that $1_{g(N)} - \varphi \circ \psi$ has essential kernel. So,

$$1_{g(N)} - \varphi \circ \psi \in J(\mathrm{End}(g(N)))$$

$\Rightarrow$ $\varphi \circ \psi$ is an isomorphism.

Similarly $\psi \circ \varphi$ is an isomorphism

Thus, $\varphi : f(M) \longrightarrow g(N)$ is an isomorphism. As $M \cong f(M)$, and $N \cong g(N)$, we have

$$M \cong N$$

Corollary: Let M, N be two modules invariant under automorphisms of their pure-injective envelopes. Let $f: M \longrightarrow N$ and $g: N \longrightarrow M$ be pure monomorphisms. Then $M \cong N$.

Question : Does the Schröder-Bernstein Problem have positive answer for those $\mathcal{X}$-automorphism-invariant modules for which the endomorphism ring of their $\mathcal{X}$-envelope is right cotorsion.

Question: Let M_1, M_2 be two flat modules invariant under automorphisms of their cotorsion envelopes such that

$$M_1 \xrightarrow{\text{mono.}} M_2 \ , \quad M_2 \xhookrightarrow{\text{mono.}} M_1$$

Is $M_1 \cong M_2$?


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